Stability in Discrete Tomography: Some Positive Results

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Abstract

The problem of reconstructing finite subsets of the integer lattice from X-rays has been studied in discrete mathematics and applied in several fields like data security, electron microscopy, medical imaging. In this paper we focus on the stability of the reconstruction problem for some special lattice sets. First we prove that if the sets are additive, then a stability result holds for very small errors. Then, we study the stability of reconstructing convex sets from both an experimental and a theoretical point of view. Numerical experiments are conducted by using linear programming and they support the conjecture that convex sets are additive with respect to a set of suitable directions. Consequently the reconstruction problem is stable. The theoretical investigation provides a stability result for convex lattice sets. This result permits to address the problem proposed by Hammer in [17].

Key words: Discrete Tomography, Stability, Linear Programming, Additivity, Convexity

1 Introduction

A *lattice set* is a non-empty finite subset of the integer lattice $\mathbb{Z}^2$. A directing vector in $\mathbb{Z}^2 \setminus \{0\}$ is called a *lattice direction*, and the *X-ray* of a lattice set E in a lattice direction p is the function X_pE giving the number of points in E on each line parallel to this direction. Discrete Tomography is the area of mathematics and computer science that deals with the inverse problem of

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reconstructing lattice sets from a finite set of X-rays. An overview on this subject highlighting the applications, the mathematical foundations, and the algorithms in Discrete Tomography is provided by the book [19].

In this paper we focus on the stability of the reconstruction problem. In literature there are few papers -that we will mention below- on this topic and a PhD thesis [1].

Informally, a problem is stable if a small perturbation of the data leads to a small change in the solution. Therefore, the stability problem is of main importance in practical applications where the X-rays are possibly affected by errors. For instance, in electron microscopy, techniques that enable to count the number of atoms lying in a line up to an error of ± 1 are known [15]. So, in case of instability, the reconstructed set can be quite different from the original one even if the error on the data is small. In general there exist more than one lattice set with given X-rays (corresponding to a null error on the data), so called tomographically equivalent, and it can be proved in a constructive way that whenever the number of X-rays is more than two there does not exist any integer l such that an arbitrary lattice set differs from any tomographically equivalent one by at most l points [18]. Therefore the requirement of uniqueness is the minimal one. In [2,1] the authors provide a modification of a generalization of the construction in [18] and show that there exist two disjoint sets uniquely determined by their X-rays when the X-rays differ by a certain quantity.

In Remark 6 we show that to obtain a stability result even with a very small error on the data the requirement of uniqueness for the sets is not enough. To this goal, we shall consider the reconstruction of lattice sets with some additional constraints.

In Section 3 we treat the stability of reconstructing additive sets. This class of sets was first introduced by Fishburn et al. in [11]. Here we just recall to the reader that additivity implies uniqueness, whereas the converse is not true. Additionally, the notion of additivity should be regarded as a property of the solutions of the linear program associated to the reconstruction problem. We prove that if the sets are additive, then a stability result holds (Proposition 7). This result permits to use linear programming to solve the reconstruction problem in a computationally efficient and effective way.

In Section 4 we study the stability of reconstructing convex lattice sets from both an experimental and a theoretical point of view. In the former, we use linear programming to deal with this problem. Experimental results suggest the conjecture that for the set of directions $\{x, y, 2x+y, -x+2y\}$, convex lattice sets are additive. This would imply that the results of Section 3 may hold to convex lattice sets. Such a stability result is in agreement with the continuous case where the reconstruction problem for convex bodies is well-posed ([25]). In the latter, the theoretical result (Proposition 18) confirms stability for convex lattice sets by exploiting the result in [25]. We use this result to address the problem proposed by Hammer in [17] concerning the reconstruction of any convex body from its (continuous) X-rays. More precisely we prove that a convex body is arbitrarily close to a convex lattice set whose X-rays are close enough to the X-rays of the convex body (see Proposition 20). In other words, this means that we can reconstruct a convex body from its discrete X-rays, if we admit a resolution and a X-ray error (due to the discretization) as small as wanted.

These results justify the use of Discrete Tomography algorithms to reconstruct continuous convex shapes from a few X-ray images. For example, we mention the reconstruction of the section of coronary arteries, supposed to be convex, from X-ray angiograms as an application. Among other studies in this direction we mention [22] where the authors propose an algorithm that approximates the convex body by a sequence of discrete objects reconstructed from X-rays.

2 The Problem

In this paper, every direction is supposed to be rational and is described by two coprime integers a and b such that any line parallel to p has an equation $ax + by =$ const. In the following we will confound the direction p with the function $(x, y) \mapsto ax + by$, and so two points A and B are on a line parallel to p if $p(A) = p(B)$. Let $p = ax + by$ and $q = cx + dy$ be such that $\gcd(a, b) = 1$, $\gcd(c, d) = 1$; then $\det(p, q) = |ad - bc|$. If S is a lattice set, then the X-ray of S in the direction p is the function $X_pS : \mathbb{Z} \to \mathbb{N}$ defined by $X_pS(k) = \text{card}\{A \in S : p(A) = k\}$.

The reconstruction problem is the task of determining any lattice set having the given X-rays. Stability concerns how sensitive is the problem to noisy data. Hence one can ask whether to small perturbations of the data correspond solutions that are close. To study the problem we define a measure for the error on the X-rays and one for the distance of two solutions. Let $\mathcal{D}$ be a set of m prescribed lattice directions with $m \geq 2$ and E, F be lattice sets. The distance between the X-rays of E and those of F is defined by

$$DX_{\mathcal{D}}(E, F) = \max_{p \in \mathcal{D}} \sum_{k \in \mathbb{Z}} |X_pE(k) - X_pF(k)|.$$

The distance between two sets is defined by:

$$\text{card}(E \triangle F) = \text{card}((E \setminus F) \cup (F \setminus E)).$$

The formulation of the problem that we consider is the following:

Problem 1 *Let E be known. Determine F maximizing* $\mathrm{card}(E \triangle F)$, *with the constraint that $DX_{\mathcal{D}}(E,F)$ is given.*

Let us introduce some definitions that we need in the following.

Definition 2 *A lattice set E is* additive with respect to $\mathcal{D}$, *or* $\mathcal{D}$-additive, *if there is a function e which gives a value $e_p(k)$ for each line $p = k$ parallel to a direction p of $\mathcal{D}$ such that for all A in $\mathbb{Z}^2$:*

$$A \in E \text{ if and only if } \sum_{p \in \mathcal{D}} e_p(p(A)) > 0.$$

This definition introduced by Fishburn et al. can be better understood with linear programming: a lattice set E is additive if it is the unique solution of the linear programming problem which looks for a fuzzy set which has the same X-rays than E.

Definition 3 *A lattice set E is* unique with respect to $\mathcal{D}$, *or* $\mathcal{D}$-unique, *if $F \subset \mathbb{Z}^2$ and $X_p E = X_p F$ for any $p \in \mathcal{D}$ imply $E = F$.*

There is an intimate relationship between these two definitions: every $\mathcal{D}$-additive set is $\mathcal{D}$-unique and the converse is true if $m = 2$ (see[11]).

As a last remark we recall that a p-line does not always intersect a q-line: indeed $\mathbb{Z}^2$ can be split in $\det(p,q)$ pq-lattice such that in each pq-lattice a p-line intersects with any q-line. Precisely a pq-lattice has the form:

$$L_i^{pq} = \{A \in \mathbb{Z}^2 \ : \ p(A) = i \pmod{\det(p,q)} \text{ and } q(A) = \kappa i \pmod{\det(p,q)}\}$$

where κ only depends on the directions p and q (see for example [8]). Moreover we denote by $\langle i,j \rangle_{pq}$ the point A such that $p(A) = i$ and $q(A) = j$. Notice that this point is in $\mathbb{Z}^2$ only if $p = i$ and $q = j$ are in the same pq-lattice.

3 Stability for Additive Sets

In this section we study the stability of reconstructing $\mathcal{D}$-additive sets. We begin to study Problem 1 with E and F verifying the constraint:
$DX_{\mathcal{D}}(E,F) \leq 1$.

In the first two lemmas additivity is not required.

The condition $DX_{\mathcal{D}}(E,F) \leq 1$ permits the X-rays of the two sets to differ by one in at most a line for each direction. Then, $p \in \mathcal{D}$ and at most an integer k_p exists such that $|X_pE(k_p) - X_pF(k_p)| = 1$ and $X_pE(k) = X_pF(k)$ for $k \neq k_p$.

Lemma 4 *If $DX_{\mathcal{D}}(E,F) \leq 1$, $p \in \mathcal{D}$, and an integer k_p exists such that $|X_pE(k_p) - X_pF(k_p)| = 1$, then for every $q \in \mathcal{D}$ there is an integer k_q such that $|X_qF(k_q) - X_qE(k_q)| = 1$ and $\langle k_p, k_q \rangle_{pq} \in \mathbb{Z}^2$.*

PROOF. Let L_i^{pq} be the pq-lattice containing the line $p = k_p$, or equivalently $k_p \in p(L_i^{pq})$ with $p(L_i^{pq}) = \{p(M) \,:\, M \in L_i^{pq}\} = \{i + \det(p,q)k \,:\, k \in \mathbb{Z}\}$. Suppose that $X_pF(k_p) - X_pE(k_p) = +1$. Thus, we have that

$$\sum_{k \in p(L_i^{pq})} X_pF(k) = 1 + \sum_{k \in p(L_i^{pq})} X_pE(k).$$

Since $\sum_k X_rF(k) = card(F)$ and $\sum_k X_rE(k) = card(E)$, for every $r \in D$, the previous identity leads to the following

$$\sum_{k \in q(L_i^{pq})} X_qF(k) = 1 + \sum_{k \in q(L_i^{pq})} X_qE(k),$$

for all q in $\mathcal{D}$. From this, the thesis easily follows. □

In the next lemma we show that all the lines with error 1 have a common point and this point is in $\mathbb{Z}^2$. In the following, we assume that $\text{card}(F) > \text{card}(E)$ and for any $p \in \mathcal{D}$ the integer k_p is as in the previous lemma.

Lemma 5 *If $DX_{\mathcal{D}}(E,F) = 1$, then a point $W \in \mathbb{Z}^2$ exists such that*

$$\begin{aligned} X_pF(k) &= X_pE(k) + 1, \;\; \textit{if}\; k = p(W) \\ X_pF(k) &= X_pE(k), \qquad \textit{otherwise} \end{aligned}$$

for all the directions p in $\mathcal{D}$.

PROOF. Let p, q and r be directions in $\mathcal{D}$ and suppose that $A = \langle k_p, k_q \rangle_{pq}$, $B = \langle k_p, k_r \rangle_{pr}$, $C = \langle k_q, k_r \rangle_{qr}$ are three distinct points. Let a, b be such that $r = ap + bq$. Thus, summing up we can write:

$$\sum_{M \in F} r(M) = a \sum_{M \in F} p(M) + b \sum_{M \in F} q(M)$$

and by grouping line by line we obtain:

$$\sum_k kX_rF(k) = a \sum_k kX_pF(k) + b \sum_k kX_qF(k).$$

We can exhibit the corresponding identity for the set E. As a result of the difference of these two identities we obtain that $k_r = ak_p + bk_q$ and so $r(A) = r(B) = r(C)$. Thus, the three points A, B and C coincide and the claim is proved. □

Remark 6 *Given any three lattice directions we can construct two sets E, F in such a way that they are (non-additive) sets of uniqueness. (For proving this we refer the reader to [11]). Figure 1, already published in [11], illustrates two such sets verifying the constraint $DX_{\mathcal{D}}(E, F) = 1$.*

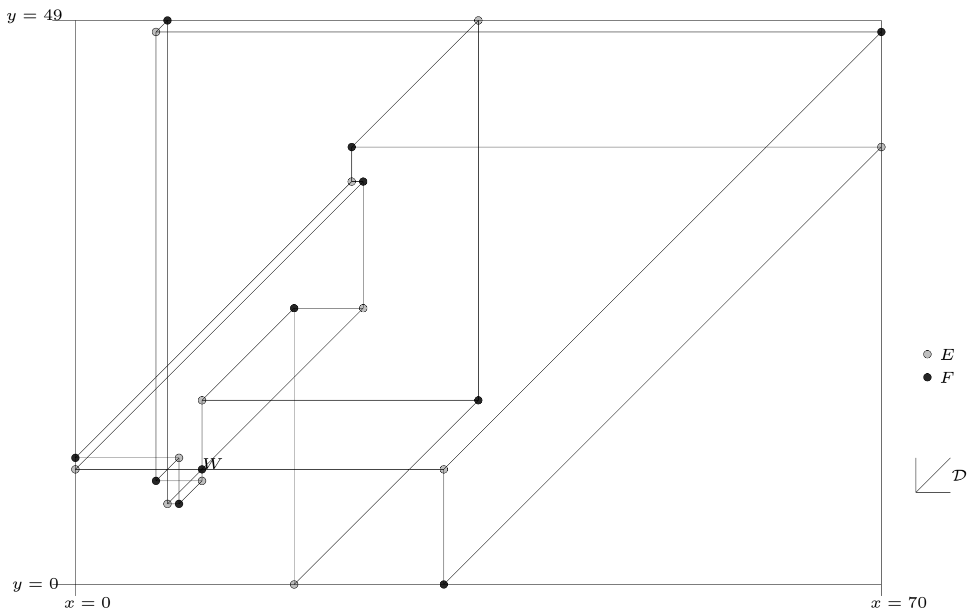

Fig. 1. E and F are non-additive sets of uniqueness such that $DX_{\mathcal{D}}(E, F) = 1$ and $E \cap F = \emptyset$.

Since uniqueness is not sufficient to have stable solutions for the reconstruction problem, we suppose that E and F are $\mathcal{D}$-additive, that is $E = \{A \in \mathbb{Z}^2 : e(A) > 0\}$ and $F = \{A \in \mathbb{Z}^2 : f(A) > 0\}$.

Proposition 7 *Let E and F be $\mathcal{D}$-additive lattice sets. If $DX_{\mathcal{D}}(E, F) = 1$, then* $\text{card}(E \triangle F) = 1$.

PROOF. Let W be as in Lemma 5. At first suppose that $W \notin E$ and let $E' = E \cup \{W\}$. For each direction p in $\mathcal{D}$ we have that $X_p E' = X_p F$. Finally, since additivity of F implies uniqueness of F, we conclude that $F = E \cup \{W\}$. On the contrary, if $W \in E$ we study the following:

$$\Phi_E = \sum_{A \in \mathbb{Z}^2} \sum_{p \in \mathcal{D}} e_p(p(A))(1_E(A) - 1_F(A)).$$

Rewriting it as

$$\sum_{A\in E}\sum_{p\in\mathcal{D}} e_p(p(A))(1_E(A)-1_F(A)) + \sum_{A\notin E}\sum_{p\in\mathcal{D}} e_p(p(A))(1_E(A)-1_F(A)),$$

we notice that $\Phi_E \geq 0$, because the additivity of E implies that if A is in E, then $e(A) > 0$ and $1_E(A) = 1$ holds, and otherwise $e(A) \leq 0$ and $1_E(A) = 0$. We can also explicit the terms X_pE and X_pF in Φ_E so obtaining that

$$\Phi_E = \sum_{k\neq p(W)}\sum_{p\in\mathcal{D}} e_p(k)(X_pE(k)-X_pF(k)) + \sum_{p\in\mathcal{D}} e_p(p(W))(X_pE(p(W))-X_pF(p(W)))$$

that is strictly less than zero. $\square$

Remark 8 *Let us notice that in the proof, additivity for F and just uniqueness for E are needed.*

If we consider the case where the error is larger than 1, we have instability even when the error is just equal to 2, if the number of lattice directions is larger than 2. More in detail, the instability follows from the result of [2, Theorem 1] because the sets constructed in the proof of [2] are actually $\mathcal{D}$-additive. Therefore we can restate it as follows:

Proposition 9 (see [2]) *For any n and a set $\mathcal{D}$ of $m \geq 3$ directions there exist E and F $\mathcal{D}$-additive such that* $\mathrm{card}(E) = \mathrm{card}(F) \geq n$, $DX_{\mathcal{D}}(E,F) = 2$ *and* $E \cap F = \emptyset$.

4 Stability for Convex Sets

In this section we study Problem 1 for convex lattice sets from both an experimental and a theoretical point of view.

Any *convex lattice set* is the intersection of a convex polygon and the integer lattice $\mathbb{Z}^2$. The definition of convex lattice set pass through that of convex polygon, and this can be used to determine results for the discrete case from the continuous case. In this way, convex lattice sets are uniquely determined by their X-rays taken in suitable sets of directions [12], and these sets of directions distinguish convex bodies [14]. So in $\mathbb{R}^2$ an analogous result holds, and additionally the reconstruction problem is stable [25]. Moreover we notice that there is a connection between additive sets and convex lattice sets, since an euclidean ball is additive with respect to two orthogonal directions ([10]). Experiments support the conjecture that convex lattice sets are additive for a suitable set of directions, and indeed they accord to Proposition 7.
In the second part, we conduct a theoretical study that confirms stability for convex lattice sets.

In this section we experimentally study the stability of the reconstruction of convex lattice sets via linear programming. Our experiments support the suspect that the results in the continuous have a correspondence in $\mathbb{Z}^2$.

For generating our test data, we consider in this section a class of lattice sets which is more general than the convex lattice sets [6].

For each point $A = (x_A, y_A) \in \mathbb{Z}^2$ the four quadrants around A are defined by the following formulas:

$$
\begin{aligned}
R_0(A) &= \{(x,y) \in \mathbb{Z}^2 \ / \ x \le x_A \text{ and } y \le y_A\},\\
R_1(A) &= \{(x,y) \in \mathbb{Z}^2 \ / \ x \ge x_A \text{ and } y \le y_A\},\\
R_2(A) &= \{(x,y) \in \mathbb{Z}^2 \ / \ x \ge x_A \text{ and } y \ge y_A\},\\
R_3(A) &= \{(x,y) \in \mathbb{Z}^2 \ / \ x \le x_A \text{ and } y \ge y_A\}.
\end{aligned}
$$

Definition 10 *A lattice set E is* Q-convex *if and only if for each $A \notin E$ there exists $i \in \{0,1,2,3\}$ such that $R_i(A) \cap E = \emptyset$.*

An example of Q-convex lattice set is given on the left part of Figure 2.

We generated 184 Q-convex lattice sets of semi-perimeter from 4 to 370 using an uniform generator ([5], inspired from [20]). Then we computed their X-rays in the set of directions $\mathcal{D} = \{x, y, 2x+y, -x+2y\}$. (These directions have been chosen because the X-rays along them uniquely determine the convex lattice sets ([12]) and they contain the horizontal and vertical directions). Given the generated set E, and its X-rays, we solved the following linear program with the software soplex which implements the simplex algorithm ([26]) for any error on the X-rays $er \in \{0,1,2,3\}$ given in input:

Maximizing

$$\sum_{(i,j)\in E} (1 - v_{i,j}) + \sum_{(i,j)\notin E} v_{i,j} \tag{1}$$

such that

$$\sum_{p(i,j)=k} v_{i,j} = X_pE(k) + er^+_{p,k} - er^-_{p,k} \tag{2}$$

$$\sum_k er^+_{p,k} + er^-_{p,k} \le er \tag{3}$$

$$0 \le v_{i,j} \le 1,\ er^+_{p,k} \ge 0,\ er^-_{p,k} \ge 0 \tag{4}$$

A variable $v_{i,j}$ is associated to the point (i,j), and constraints (2) and (3) ensure that the error on the X-rays is $\le er$. Notice that, because of the objective function, solving this problem with $v_{i,j} \in \mathbb{Z}$ would permit to exactly find

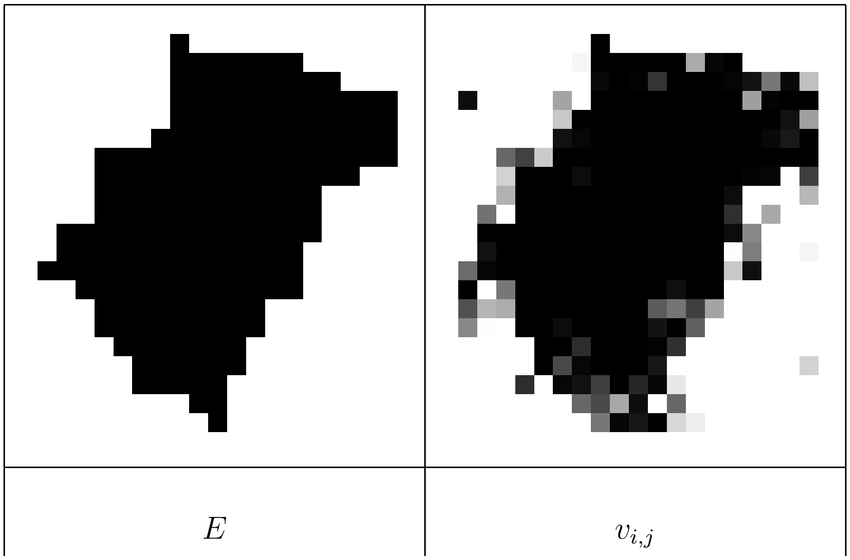

Fig. 2. A Q-convex lattice set E and the corresponding extremal values of $v_{i,j}$ for $er = 3$. In this case we have $\text{card}(E) = 200$ and $\sum_{(i,j)\in E}(1 - v_{i,j}) + \sum_{(i,j)\in E^c} v_{i,j} = 33.7$.

the maximum of $\text{card}(E \triangle F)$ where F describes *all* the lattice sets such that $DX_{\mathcal{D}}(E, F) \leq er$. Indeed if $v_{i,j} = 1$, then (i, j) belongs to any set F. Unfortunately, integer-linear-program is an $\mathbb{NP}$-hard problem, and hence we solved the *relaxed* problem where the unknown variables can be fractional: this computation provides an upper bound to $\text{card}(E \triangle F)$. Figure 2 illustrates (on the right side-hand) a solution of the linear programming for $\text{card}(E) = 200$ and $er = 3$. The different grey-scale colors of the squares correspond to different values of $v_{i,j}$.

The complete results are summarized in Figures 3 and 4. In Figure 3 the upper bound to $\text{card}(E \triangle F)$ is divided by $\text{card}(E)$, so that each value gives an upper bound to the relative distance from a given set. Moreover the black squares show the values of the maximum of the quantity (1) when the constraints (2),(3) are replaced by $X_pE(k) - 1 \leq \sum_{p((i,j))=k} v_{i,j} \leq X_pE(k) + 1$: these values give an upper bound to $\text{card}(E \triangle F)$ when $DX'_{\mathcal{D}}(E, F) = \max_{p\in\mathcal{D}} \max_{k\in\mathbb{Z}} |X_pE(k) - X_pF(k)| = 1$.

Here we report on the experimental results.

- If $DX_{\mathcal{D}}(E, F) = 0$, then we always found a null relative distance. In other words, according to our experiments every Q-convex lattice set is $\mathcal{D}$-additive. In fact this property was first conjectured by L. Thorens ([24]) (with additivity replaced by uniqueness), and can be seen as a variant of Conjecture 4.6 of [3] and Theorem 5.7 of [12]. We can set out the conjecture as follows: **Conjecture 11** *If $\mathcal{D}$ is a set of directions which contains $\{x, y\}$, such that all the directions are not in the same quadrant and they uniquely determine the convex lattice sets, then every Q-convex lattice set is $\mathcal{D}$-additive.*

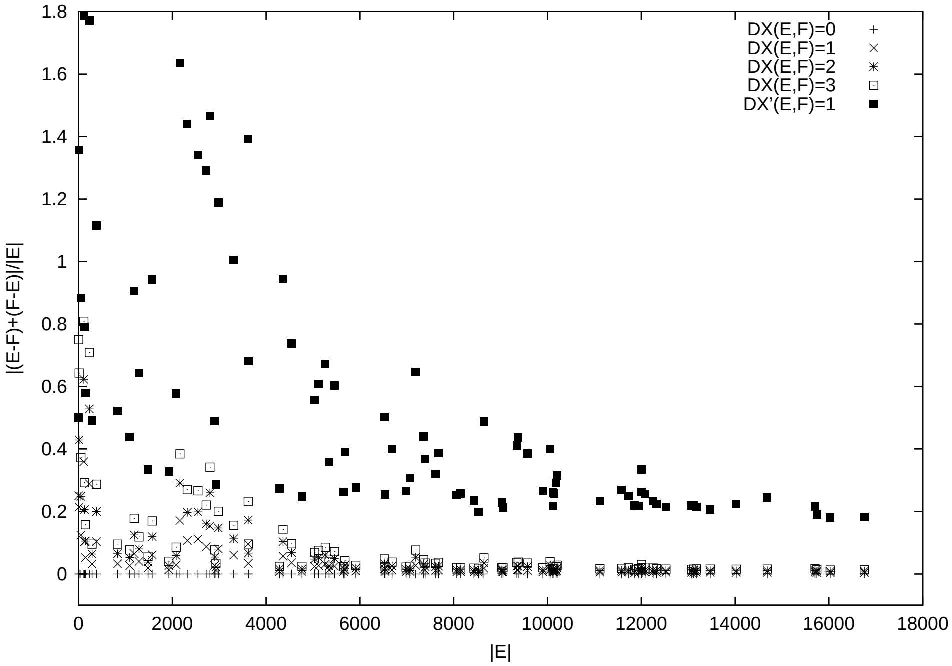

Fig. 3. An upper bound to $\frac{\text{card}(E \triangle F)}{\text{card}(E)}$ for the Q-convex generated sets. (Only 40 % of the 184 generated sets have been represented for readability)

Notice that the property about the quadrants is necessary because there is a counter-example with $\mathcal{D} = \{x, y, x+y, x+5y\}$.

- If $DX_{\mathcal{D}}(E,F) = er$, the relative distance looks to converge to zero as $\text{card}(E)$ grows. If we divide by $\sqrt{\text{card}(E)}$ instead of $\text{card}(E)$, this ratio seems to be bounded so that in *average* $\text{card}(E \triangle F) = O(\sqrt{\text{card}(E)})$ according to our experiments (see Figure 4). It must be noticed that when $er = 1$, the maximum distance of any two sets is always 1 for the generated cases according to the result of Proposition 7. Since the theoretical result holds for additive sets, the experiments could be interpreted as a further evidence of the conjecture.
- If $DX'_{\mathcal{D}}(E,F) = 1$, then the relative distance does not seem to converge to zero, but the computed values are only upper bounds, that is, we do not know if the fractional values mirror instability or they are just an artifact introduced by relaxing the integral constraints of the problem.

4.2 Theoretical results

In this section we first exploit a stability result for convex bodies [25] to deal with the corresponding problem for convex lattice sets and then we use this result to show that it is possible, at least theoretically, to reconstruct convex bodies from X-rays, by means of Discrete Tomography.

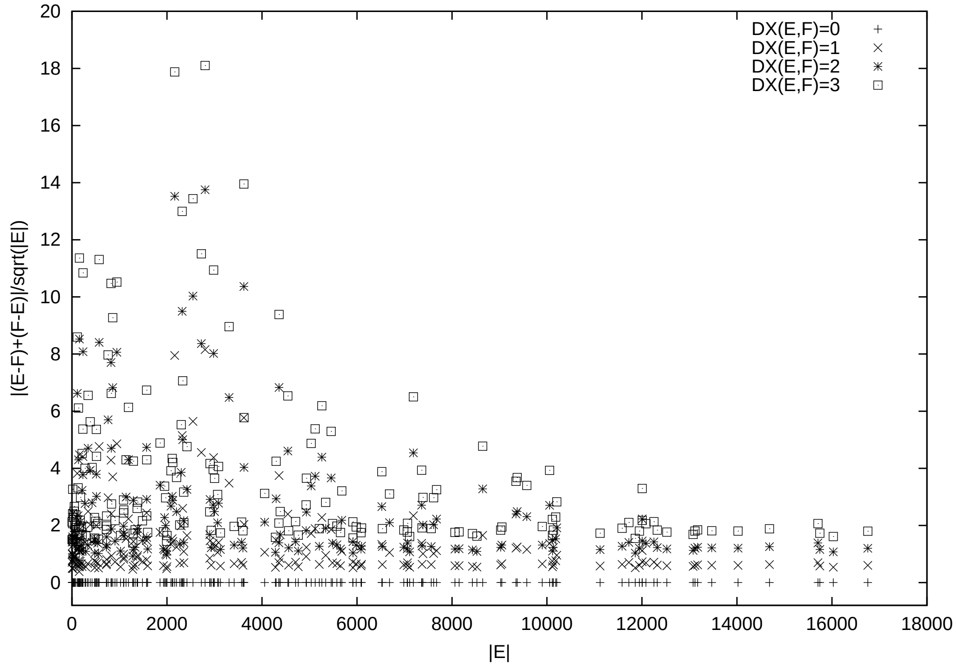

Fig. 4. An upper bound to $\frac{\operatorname{card}(E \triangle F)}{\sqrt{\operatorname{card}(E)}}$ for the 184 generated Q-convex lattice sets

4.2.1 Preliminaries

A *convex body* is a compact convex subset of $\mathbb{R}^2$ with non-empty interior. We denote the set of all the convex bodies by $\mathcal{K}_*$. The X-ray X_pU of the convex body U in direction p is the function giving the length of each chord of U parallel to p. More precisely $X_pU(\alpha)$ is the length of the intersection of U with the line $p = \alpha$. The *Steiner symmetral* $S_p(U)$ of U in direction p is the closure of the union of all open segments on lines parallel to p of the same length as X_pU centered about a fixed line orthogonal to p. So the Steiner symmetral $S_p(U)$ and the X-ray X_pU contains exactly the same information.

Definition 12 *A set of directions $\mathcal{D}$ is a* Gardner-McMullen set of directions *if any convex body is characterized by all its X-rays in the directions of $\mathcal{D}$.*

We recall a result of [13, Proposition 6.1,Theorem 4.5]:

Theorem 13 (Gardner-Gritzmann) *A set $\mathcal{D} = \{p_1, p_2, p_3, p_4\}$ of four* lattice *directions is a Gardner-McMullen set of directions if and only if the cross-ratio of the directions arranged in order of increasing angle with the positive x-axis is not in $\{\frac{4}{3}, \frac{3}{2}, 2, 3, 4\}$.*

Example 14 *This theorem implies that the set $\mathcal{D} = \{x, y, 2x + y, -x + 2y\}$ is a Gardner-Gritzmann set of directions.*

In the following we suppose that $\mathcal{D} = \{p_1, p_2, p_3, p_4\}$ is a Gardner-McMullen set of four directions. So the mapping $\mu : \mathcal{K}_* \mapsto (S_{p_1}(U), S_{p_2}(U), S_{p_3}(U), S_{p_4}(U))$ is injective.

Let $\mathcal{K}_*$ be endowed with Nikodym's distance:

$$d_N(U, V) = m(U \triangle V),$$

where $m(U)$ denotes the Lebesgue measure on $\mathbb{R}^2$. Now are we in place to state the stability result for convex bodies (see Theorem of [25, section 3.1]):

if $\mathcal{K}_*$ is endowed with the topology induced by the Nikodym's distance, μ is continuous and continuously invertible from $\mu(\mathcal{K}_*)$.

We shall reformulate this theorem. Consider the map $\sigma_{\mathcal{D}} : U \mapsto (X_p U)_{p\in\mathcal{D}}$; if $\mathcal{D}$ is a Gardner-McMullen set of direction, then $\sigma_{\mathcal{D}}(U)$ is injective. Let $\mathcal{X}_{\mathcal{D}}$ be the range of $\sigma_{\mathcal{D}}$. We endow $\mathcal{X}_{\mathcal{D}}$ with the following distance:

$$d_{\mathcal{X}}((f_p)_{p\in\mathcal{D}}, (g_p)_{p\in\mathcal{D}}) = \max_{p\in\mathcal{D}} \int_{-\infty}^{+\infty} \frac{|f_p(\alpha) - g_p(\alpha)|}{\sqrt{a_p^2 + b_p^2}} \mathrm{d}\alpha,$$

where $(f_p)_{p\in\mathcal{D}}, (g_p)_{p\in\mathcal{D}}$ are in $\mathcal{X}_{\mathcal{D}}$, a_p and b_p are defined by $p(x, y) = a_p x + b_p y$. (Notice that each integral in the definition of this distance corresponds exactly to the Nikodym distance if X-rays are considered as Steiner symmetral.) We also use the notation $d_{\mathcal{X}_{\mathcal{D}}}(U, V) = d_{\mathcal{X}}(\sigma_{\mathcal{D}}(U), \sigma_{\mathcal{D}}(V))$.

The theorem of [25, section 3.1] can be rewritten as follows:

Theorem 15 (Volčič) *Let $\mathcal{D}$ be a Gardner-McMullen set of four lattice directions the inverse $\sigma_{\mathcal{D}}^{-1}$ of the function $\sigma_{\mathcal{D}}$ is a continuous function from $\mathcal{X}_{\mathcal{D}}$ to $\mathcal{K}_*$.*

For any bounded set $E \subset \mathbb{R}^2$ we define $R_{\max}(E) = \max_{M\in E} \|M\|$ where $\|\cdot\|$ is the euclidean norm. The set $\mathcal{K}_\varepsilon^1 = \{U \in \mathcal{K}_* \ : \ R_{\max}(U) \leq 1 \text{ and } m(U) \geq \varepsilon\}$ is a compact subset of $\mathcal{K}_*$; it follows that $\sigma_{\mathcal{D}}(\mathcal{K}_\varepsilon^1)$ is a compact subset of $\mathcal{X}_{\mathcal{D}}$ and so the function $\sigma_{\mathcal{D}}^{-1}$ restricted to $\sigma_{\mathcal{D}}(\mathcal{K}_\varepsilon^1)$ is uniformly continuous. So we can give a more precise formulation of the previous theorem:

Corollary 16 *Let $\mathcal{D}$ be a Gardner-McMullen set of four lattice directions. For any $\varepsilon > 0$ there exists $\eta > 0$ such that any $U, V \in \mathcal{K}_\varepsilon^1$ satisfy:*

$$d_{\mathcal{X}_{\mathcal{D}}}(U, V) < \eta \Longrightarrow d_N(U, V) < \varepsilon.$$

4.2.2 A stability result for convex lattice sets

In this section $\mathcal{D}$ is a Gardner-McMullen set of four lattice directions. Since a Gardner-McMullen set of lattice directions uniquely determines convex lattice sets [12], we use the result enunciated in Corollary 16 to get a stability result for convex lattice sets.

At first we need a lemma which is a direct consequence of Pick's theorem ([23,16]). We recall that a lattice polygon is a polygon whose vertices are in $\mathbb{Z}^2$, and a simple polygon is a polygon whose edges have a non-empty intersection only if they are consecutive.

Lemma 17 *If $P \subset \mathbb{R}^2$ is simple lattice polygon which is not a segment, then* $\mathrm{card}(P \cap \mathbb{Z}^2) \leq 2m(P) + 2$.

As both the distance of two lattice sets, $\mathrm{card}(E \triangle F)$, and the distance of the corresponding X-rays, $DX(E, F)$, have integer values, making these quantities tend to zero has no sense, and hence, to apply Corollary 16, the *relative* distances $\frac{\mathrm{card}(E \triangle F)}{(\max(R_{\max}(E), R_{\max}(F)))^2}$ and $\frac{DX(E,F)}{(\max(R_{\max}(E), R_{\max}(F)))^2}$ will be used. The following proposition gives an upper bound to the relative distance of two lattice sets depending on the relative distance of their X-rays. More precisely, the relative distance of two lattice sets is smaller than a multiple of the inverse of a quantity expressing the maximum size of the sets into consideration plus a quantity which can be arbitrarily small, if the relative distance of the X-rays is small enough.

We suppose that each direction p of $\mathcal{D}$ has the form $p((x, y)) = a_p x + b_p y$ with a_p, b_p integer.

Proposition 18 *For any $\varepsilon > 0$ and $K > 1$, there exist $\eta > 0, M > 0$ such that any lattice convex non-segment sets E and F such that $\frac{\mathrm{card}(E)}{(R_{\max}(E))^2}, \frac{\mathrm{card}(F)}{(R_{\max}(F))^2} \geq \varepsilon$, $R_{\max}(E), R_{\max}(F) \geq M$, $\frac{1}{K} \leq \frac{R_{\max}(E)}{R_{\max}(F)} \leq K$ satisfy:*

$$\frac{DX(E,F)}{(\max(R_{\max}(E), R_{\max}(F)))^2} < \eta \Longrightarrow \frac{\mathrm{card}(E \triangle F)}{(\max(R_{\max}(E), R_{\max}(F)))^2} < \varepsilon + \frac{17}{\max(R_{\max}(E), R_{\max}(F))}$$

PROOF. We define $\varepsilon_c = \frac{\varepsilon}{2}$. Let η_c given by Corollary 16 applied to ε_c. We take M such that $\frac{6}{M} \leq \frac{\eta_c}{2}$, $\frac{1}{(KM)^2} \leq \varepsilon_c$ and $M \geq 8$. So we suppose that E and F are sets which satisfy the conditions of the proposition.

Let us consider the number $N = \max(R_{\max}(E), R_{\max}(F))$ and the sets $E_c =$

$\frac{1}{N}\text{conv}(E)$, $F_c = \frac{1}{N}\text{conv}(F)$.

The sets E_c and F_c are convex polygons of $\mathbb{R}^2$, and since they are not segments, E_c and F_c are convex bodies. Additionally, they are simple *lattice* polygons being their vertices in $(\mathbb{Z}/N)^2$. By Lemma 17 applied to $P = N \cdot E_c$ we have $m(E_c) \geq \frac{1}{N^2}(\frac{\text{card}(E)}{2} - 1)$. So:

$$\begin{aligned} m(E_c) &\geq \frac{1}{N^2}(\frac{\text{card}(E)}{2} - 1) \\ &\geq \frac{\text{card}(E)}{2(KR_{\max}(E))^2} - \frac{1}{(KR_{\max}(E))^2} \\ &\geq \frac{\varepsilon}{2K^2} - \frac{1}{(KM)^2} \\ &\geq 2\varepsilon_c - \varepsilon_c = \varepsilon_c \end{aligned}$$

Similarly $m(F_c) \geq \varepsilon_c$. So $E_c, F_c \in \mathcal{K}^1_\varepsilon$.

Now we suppose that $\frac{DX(E,F)}{N^2} < \eta$ with $\eta = \frac{\eta_c}{4}$ and we estimate $d_{\mathcal{X}_\mathcal{D}}(E_c, F_c)$. We have that:

$$\frac{X_pE(n) - 1}{N} \leq X_pE_c(\frac{n}{N}) \leq \frac{X_pE(n) + 1}{N}$$

$$\frac{X_pF(n) - 1}{N} \leq X_pF_c(\frac{n}{N}) \leq \frac{X_pF(n) + 1}{N}$$

so that

$$|X_pE_c(\frac{n}{N}) - X_pF_c(\frac{n}{N})| \leq \frac{|X_pE(n) - X_pF(n)| + 2}{N}.$$

Since

$$|X_pE_c(\alpha) - X_pF_c(\alpha)| \leq \max(|X_pE_c(\frac{\lfloor \alpha N \rfloor}{N}) - X_pF_c(\frac{\lfloor \alpha N \rfloor}{N})|, |X_pE(\frac{\lceil \alpha N \rceil}{N}) - X_pF(\frac{\lceil \alpha N \rceil}{N})|)$$

we get

$$\begin{aligned} \int_{-\infty}^{+\infty} |X_pE_c(\alpha) - X_pF_c(\alpha)| \mathrm{d}\alpha &\leq \frac{2}{N} \sum_{n=\lceil -N\sqrt{a_p^2+b_p^2} \rceil}^{\lfloor N\sqrt{a_p^2+b_p^2} \rfloor} \frac{|X_pE(n) - X_pF(n)| + 2}{N} \\ &= \frac{2}{N^2} \sum_{n=-\infty}^{+\infty} |X_pE(n) - X_pF(n)| + \frac{2(2N\sqrt{a_p^2 + b_p^2} + 1)}{N^2}. \end{aligned}$$

Finally, a_p and b_p are integer and, so $\sqrt{a_p^2+b_p^2} \geq 1$, and we conclude that

$$\begin{aligned} d_{\mathcal{X}_\mathcal{D}}(E_c, F_c) &\leq \frac{2}{N^2} DX(E,F) + \frac{6}{N} \\ &\leq 2\eta + \frac{6}{M} \\ &\leq \frac{\eta_c}{2} + \frac{\eta_c}{2} = \eta_c. \end{aligned}$$

By Corollary 16 we have that $d_N(E_c, F_c) \leq \varepsilon_c \leq \frac{\varepsilon}{2}$.

The symmetric difference $E_c \triangle F_c$ is the union of components of $E_c \setminus F_c$ and of $F_c \setminus E_c$. Let C_j denotes the closure of the jth component and $E_c \triangle F_c = \cup_{j=1}^k C_j$.

Each component C_j is a simple polygon $(S_1 A_1 A_2 A_3 \dots A_{m_1} S_2 B_1 B_2 \dots B_{m_2})$ where $A_1, \dots A_{m_1}$ and $B_1, \dots B_{m_2}$ are consecutive vertices of E_c and F_c, respectively, and S_1, S_2 are intersections of an edge of E_c with an edge of F_c. The component C_j contains at least one vertex of E_c or one of F_c so $m_1 + m_2 > 0$.

For each j we define the lattice polygon C'_j as follows:

- If $m_1 = 0$ or $m_2 = 0$ then C'_j is $\mathrm{conv}(C_j \cap (\mathbb{Z}/N)^2)$. (In this case C_j is convex.)
- If $m_1 > 0$ and $m_2 > 0$ then C'_j is the union of the three following polygons:
  - · $\mathrm{conv}((S_1 A_1 B_1) \cap (\mathbb{Z}/N)^2)$
  - · the lattice polygon $(A_1 A_2 \dots A_{m_1} B_1 B_2 \dots B_{m_2})$
  - · $\mathrm{conv}((S_2 A_{m_1} B_{m_2}) \cap (\mathbb{Z}/N)^2)$

This polygon C'_j is included in C_j and is a simple polygon whose vertices are all in $(\mathbb{Z}/N)^2$, so if it is not a segment, by Lemma 17 we have $\mathrm{card}(C_j \cap (\mathbb{Z}/N)^2) = \mathrm{card}(C'_j \cap (\mathbb{Z}/N)^2) \leq 2m(C'_j) + 2 \leq 2m(C_j) + 2$.

Let l be the number of C' components that are not a segment and
$S = \sum_{\substack{j=1 \\ C_j \text{ is a segment}}}^{k} \mathrm{card}(C'_j \cap (\mathbb{Z}/N)^2)$.

So we have (see also Figure 5):

$$\begin{aligned} \mathrm{card}(E \triangle F) &= \sum_{j=1}^{k} \mathrm{card}(C_j \cap (\mathbb{Z}/N)^2) \\ &\leq \sum_{\substack{j=1 \\ C_j \text{ is not segment}}}^{k} (2m(C_j) + 2) + \sum_{\substack{j=1 \\ C_j \text{ is a segment}}}^{k} \mathrm{card}(C'_j \cap (\mathbb{Z}/N)^2) \\ &\leq 2 d_N(E_c, F_c) + 2l + S \end{aligned}$$

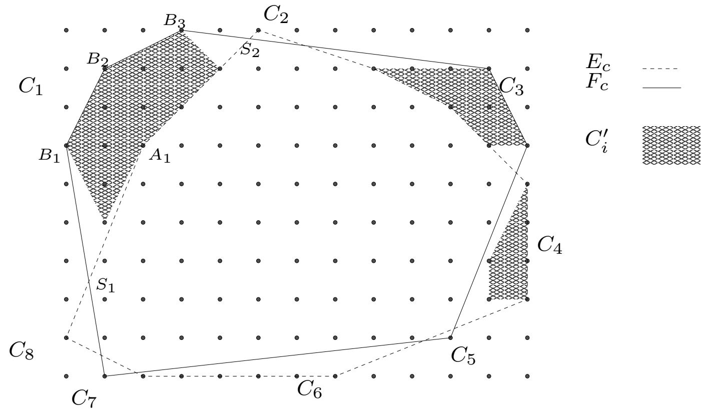

Fig. 5. The components C_i and C'_i of the symmetrical difference of two lattice convex polygons E_c and F_c. Only the points S_k, A_k, B_k of the component C_1 are annotated. The sets C'_2, C'_5, C'_7, C'_8 contain only one point and C'_6 is a segment.

The vertices of E_c and F_c are in $(\mathbb{Z}/N)^2 \cap [-1,1]^2$ so the polygons E_c and F_c have less than $4(2N+1)$ vertices whose coordinates multiplied by N are in the square $[-N,N]^2 \cap \mathbb{Z}^2$. We deduce that $l + S \leq 4(2N+1)$ and so $2l + S \leq 8(2N+1)$. Moreover $d_N(E_c, F_c) \leq \frac{\varepsilon}{2}$ so $\text{card}(E \triangle F) \leq \varepsilon N^2 + 8(2N+1) = \varepsilon N^2 + 16N + 8$. We have supposed that $N \geq M \geq 8$ so finally $\text{card}(E \triangle F) \leq \varepsilon N^2 + 17N$.

□

This upper bound overestimates the symmetric difference because we actually count also points of the border of $E_c \cap F_c$.

4.2.3 Reconstruction of a convex body from noisy discrete X-rays

In this section we always suppose that $\mathcal{D}$ is a Gardner-McMullen set of directions. If F is a convex body then we know that it is completely determined by its continuous X-rays in $\mathcal{D}$. The aim of this section is to show that it is possible, in theory, to reconstruct F by using Discrete Tomography.

Let us fix an integer n, and let E_n an approximation of F to the resolution $\frac{1}{n}$.

But as we do not know F, the only assertions about E_n concerns the distance of the discrete X-rays of E_n and the continuous X-rays of F. Proposition 20 claims that assertions which only consider the X-rays of E_n exist such that the set E_n converges, in a certain sense, to F when n tends to infinity.

We start with an easy lemma which will be useful in the following:

Lemma 19 *If E is any bounded subset of $\mathbb{R}^2$ and p, p' are two directions then*

$$\max(\frac{|\alpha_1|}{\|p\|}, \frac{|\alpha_2|}{\|p\|}, \frac{|\alpha_1'|}{\|p'\|}, \frac{|\alpha_2'|}{\|p'\|}) \leq R_{\max}(E)$$
$$\leq \max(\|\langle \alpha_1, \alpha_1' \rangle_{pp'}\|, \|\langle \alpha_1, \alpha_2' \rangle_{pp'}\|, \|\langle \alpha_2, \alpha_1' \rangle_{pp'}\|, \|\langle \alpha_2, \alpha_2' \rangle_{pp'}\|)$$

where $\alpha_1 = \inf_{z\in E} p(z),\ \alpha_2 = \sup_{z\in E} p(z),\ \alpha_1' = \inf_{z\in E} p'(z),\ \alpha_2' = \sup_{z\in E} p'(z)$.

Proposition 20 *Let F be a convex body, and $(E_n)_{n\in\mathbb{N}}$ a sequence of non-segment convex lattice sets such that for each $p \in \mathcal{D}$ there hold:*

$$\frac{1}{n}\max\{k \in \mathbb{Z} \ :\ X_pE_n(k) \neq 0\} \underset{n\to\infty}{\longrightarrow} \sup\{\alpha \in \mathbb{R} \ :\ X_pF(\alpha) \neq 0\} \qquad (5)$$

$$\frac{1}{n}\min\{k \in \mathbb{Z} \ :\ X_pE_n(k) \neq 0\} \underset{n\to\infty}{\longrightarrow} \inf\{\alpha \in \mathbb{R} \ :\ X_pF(\alpha) \neq 0\} \qquad (6)$$

$$\frac{1}{n}\max_{p\in\mathcal{D}}\sum_{k\in\mathbb{Z}}|\frac{X_pE_n(k)}{n} - X_pF(\frac{k}{n})| \underset{n\to\infty}{\longrightarrow} 0 \qquad (7)$$

then

$$\frac{1}{n^2}\mathrm{card}(E_n \triangle (nF \cap \mathbb{Z}^2)) \underset{n\to\infty}{\longrightarrow} 0.$$

PROOF. Let $(F_n)_{n\in\mathbb{N}}$ be the sequence of convex lattice sets, defined by $F_n = nF\cap\mathbb{Z}^2$. To prove this proposition we are going to show that an integer N exists such that for $n > N$ the sets E_n and F_n verify the conditions of Proposition 18. The thesis follows by applying the proposition.

At first we derive some conditions we need to our goal.

Since $\frac{\mathrm{card}(F_n)}{n^2} \underset{n\to\infty}{\longrightarrow} m(F)$ and $\frac{R_{\max}(F_n)}{n} \underset{n\to\infty}{\longrightarrow} R_{\max}(F)$, it follows that

$$\frac{\mathrm{card}(F_n)}{(R_{\max}(F_n))^2} \underset{n\to\infty}{\longrightarrow} \frac{m(F)}{(R_{\max}(F))^2} > 0. \qquad (8)$$

We have

$$\frac{X_pF_n(k) - 1}{n} \leq X_pF(\frac{k}{n}) \leq \frac{X_pF_n(k) + 1}{n},$$

and, hence by condition (7)

$$\frac{1}{n^2}DX(E_n, F_n) \underset{n\to\infty}{\longrightarrow} 0. \qquad (9)$$

As a consequence of this and $|\,\mathrm{card}(E_n) - \mathrm{card}(F_n)\,| \leq DX(E_n, F_n)$, we obtain that $\frac{\mathrm{card}(E_n)-\mathrm{card}(F_n)}{n^2} \underset{n\to\infty}{\longrightarrow} 0$.

We choose arbitrarily two directions p, p' of $\mathcal{D}$. Let $\alpha_1 = \min_{z\in F} p(z)$, $\alpha_2 = \max_{z\in F} p(z)$, $\alpha_1' = \min_{z\in F} p'(z)$, $\alpha_2' = \max_{z\in F} p'(z)$. By Lemma 19 applied to E_n, the conditions (5),(6) and the continuity of the function $(\alpha,\alpha') \to \langle\alpha,\alpha'\rangle_{pp'}$, there exists an integer N_1 such that for $n > N_1$ we have: $M_1 \leq \frac{R_{\max}(E_n)}{n} \leq M_2$ with $M_1 = \frac{1}{2}\max(\frac{|\alpha_1|}{\|p\|}, \frac{|\alpha_2|}{\|p\|}, \frac{|\alpha_1'|}{\|p'\|}, \frac{|\alpha_2'|}{\|p'\|})$ and $M_2 = 2\max(\|\langle\alpha_1,\alpha_1'\rangle_{pp'}\|, \|\langle\alpha_1,\alpha_2'\rangle_{pp'}\|, \|\langle\alpha_2,\alpha_1'\rangle_{pp'}\|, \|\langle\alpha_2,\alpha_2'\rangle_{pp'}\|)$.Thus, by this and the previous deduction, we get

$$\frac{\text{card}(E_n)}{(R_{\max}(E_n))^2} \geq \frac{\text{card}(E_n)}{(nM_2)^2} \underset{n\to\infty}{\longrightarrow} \frac{m(F)}{(M_2)^2} > 0. \tag{10}$$

Moreover, an integer $N_2 > N_1$ exists such that for $n > N_2$ there holds: $\frac{M_1}{2R_{\max}(F)} \leq \frac{R_{\max}(E_n)}{R_{\max}(F_n)} \leq \frac{2M_2}{R_{\max}(F)}$.

Now we are going to use these properties to show that we can apply Proposition 18 to E_n and F_n. To prove the thesis, we have to find, for any $\varepsilon > 0$, an N such that $\frac{1}{n^2}\text{card}(E_n \triangle (nF\cap\mathbb{Z}^2)) \leq \varepsilon$ for $n > N$. Let $K = \max(\frac{2M_2}{R_{\max}(F)}, \frac{2R_{\max}(F)}{M_1})$ and $\varepsilon' = \frac{\varepsilon}{2(KR_{\max}(F))^2}$. Without loss of generality let us suppose that $0 < \varepsilon' < \frac{m(F)}{2(R_{\max}(F))^2}, \frac{m(F)}{2(M_2)^2}$.

- We have that $\frac{1}{K} \leq \frac{R_{\max}(E_n)}{R_{\max}(F_n)} \leq K$ for $n > N_2$.
- By the conditions (8) and (10) there exists an integer N_3 such that for $n > N_3$ we have $\frac{\text{card}(E_n)}{(R_{\max}(E_n))^2}, \frac{\text{card}(F_n)}{(R_{\max}(F_n))^2} \geq \min(\frac{m(F)}{2(R_{\max}(F))^2}, \frac{m(F)}{2(M_2)^2}) \geq \varepsilon'$. Hence $\frac{\text{card}(E_n)}{(R_{\max}(E_n))^2}, \frac{\text{card}(F_n)}{(R_{\max}(F_n))^2} \geq \varepsilon'$ for $n > N_3$.
- For any fixed $M > 0$ there exists an integer N_4 such that for $n > N_4$ we have $R_{\max}(E_n) \geq M$ and $R_{\max}(F_n) \geq M$.
- For any fixed $\eta > 0$, by property (9), an integer N_5 exists such that $\frac{1}{n^2}DX(E_n, F_n) < \eta(\frac{R_{\max}(F)}{2K})^2$ for $n > N_5$.
- There exists an integer N_6 such that $\frac{R_{\max}(F_n)}{n} \geq \frac{R_{\max}(F)}{2}$ for $n > N_6$.

Now we suppose that $n > N = \max(N_2, N_3, N_4, N_5, N_6)$. Then the sets E_n and F_n satisfy the conditions of Proposition 18 (with ε' instead of ε and η and M chosen as in the proposition).

Therefore we have:

$$\frac{DX(E_n, F_n)}{(\max(R_{\max}(E_n), R_{\max}(F_n)))^2} < \eta \Longrightarrow$$
$$\frac{\text{card}(E_n \triangle F_n)}{(\max(R_{\max}(E_n), R_{\max}(F_n)))^2} < \varepsilon' + \frac{17}{\max(R_{\max}(E_n), R_{\max}(F_n))}. \tag{11}$$

By the definition of F_n, $F_n \subset nF$ and so $R_{\max}(F_n) \leq nR_{\max}(F)$. Moreover, $R_{\max}(E_n) \leq KR_{\max}(F_n)$. It follows that $\max(R_{\max}(E_n), R_{\max}(F_n)) \leq$

$KR_{\max}(F_n) \leq KnR_{\max}(F)$, since $K > 1$.

We have $R_{\max}(E_n) \geq \frac{R_{\max}(F_n)}{K}$ and by definition of N_6, $R_{\max}(F_n) \geq \frac{R_{\max}(F)}{2}$, so $\max(R_{\max}(E_n), R_{\max}(F_n))) \geq \frac{R_{\max}(F_n)}{K} \geq \frac{nR_{\max}(F)}{2K}$. Then $\frac{DX(E_n,F_n)}{(\max(R_{\max}(E_n),R_{\max}(F_n)))^2} < \frac{n^2\eta(\frac{R_{\max}(F)}{2K})^2}{(\frac{nR_{\max}(F)}{2K})^2} = \eta$, so the premise of the implication of (11) is true, therefore:

$$\begin{aligned}
\mathrm{card}(E_n \triangle F_n) &\leq \varepsilon'(R_{\max}(E_n), R_{\max}(F_n)))^2 + 17\max(R_{\max}(E_n), R_{\max}(F_n)) \\
&\leq \varepsilon'(KnR_{\max}(F))^2 + 17(KnR_{\max}(F)) \\
&\leq n^2\frac{\varepsilon}{2} + 17KR_{\max}(F)n \\
&\leq n^2\frac{\varepsilon}{2} + n^2\frac{\varepsilon}{2} \qquad \text{for } n > \frac{34KR_{\max}(F)}{\varepsilon} \\
&= \varepsilon n^2
\end{aligned}$$

□

5 Conclusion and open questions

In this paper we have proved two positive stability results in Discrete Tomography, the first one concerning additive sets and very small errors, while the second one regarding convex sets and a relative error which tends to zero. As a consequence of the second result Proposition 20 is particularly interesting because it can be used to solve Hammer's X-ray problem: consider a convex body $E \subset \mathbb{R}^2$ which is only known from X-rays along a Gardner-McMullen set $\mathcal{D}$ of directions. Then one knows that for any fixed n there is a convex lattice set $E_n \subset \mathbb{Z}^2$ such that for every direction $p \in \mathcal{D}$:

$$\left\lfloor \frac{nX_pE(\frac{i}{n})}{\sqrt{a_p^2 + b_p^2}} \right\rfloor \leq X_pE_n(i) \leq \left\lfloor \frac{nX_pE(\frac{i}{n})}{\sqrt{a_p^2 + b_p^2}} \right\rfloor + 1 \qquad (12)$$

(take $E_n = nE \cap \mathbb{Z}^2$.) It is easy to see that condition (12) implies that (E_n) satisfies the conditions of Proposition (20) and so the sequence of sets $\frac{1}{n}E_n$ tends to the searched set E. Thus, this result could be applied each time a binary convex shape is asked from a few X-ray images, if we are able to reconstruct a convex lattice set from approximative X-rays like (12). Unfortunately, algorithms are known in the exact case ([6]) only or for more general classes than convex lattice sets ([4,7]).

Proposition 20 states that the distance between the lattice set and the searched set tends to zero when the X-ray discretization error tends to zero, but a quantitative result would give much more information. Indeed we are interested in the following question: if (E_n) is a sequence of sets satisfying (12), is there a constant c such that $\mathrm{card}(E_n \triangle (nE \cap \mathbb{Z}^2)) \leq cn$?

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